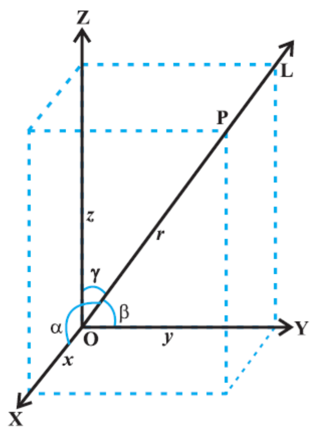
Class XII: Mathematics
Chapter 10: Three Dimensional Geometry

Chapter Notes

Key Concepts

1. The angles α, β and γ which a directed line L through the origin makes with the x , y and z axes respectively are called direction angles.

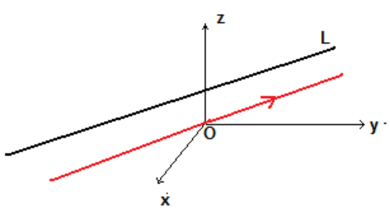


If the direction of line L is reversed then direction angles will be $\pi-\alpha$, $\pi-\beta$, $\pi-\gamma$.

2. If a directed line L passes through the origin and makes angles α , β and γ respectively with the x , y and z axes respectively , then
$\lambda = \cos\alpha$, m= $\cos\beta$ and n= $\cos\gamma$ are called direction cosines of line L.

3. For a given line to have unique set of direction cosines take a directed line.

4. The direction cosines of the directed line which does not pass through the origin can be obtained by drawing a line parallel to it and passing through the origin



5.Any three numbers which are proportional to the direction cosines of the line are called direction ratios. If λ , m, n are the direction cosines and a, b,c are the direction ratios then λ=ka , m=kb, n=kc where k is any non zero real number.
6. For any line there are an infinite number of direction ratios.

7. Direction ratios of the line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ may be taken as ,

$x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$ or $x_1 - x_2$, $y_1 - y_2$, $z_1 - z_2$

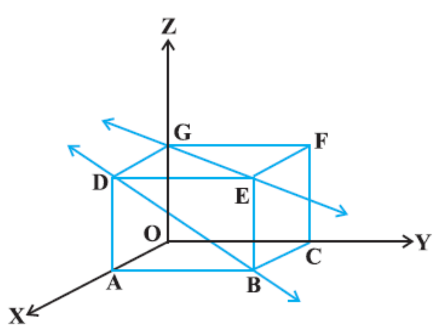
8. Direction cosines of x-axis are cos0, cos 90, cos90 i.e. 1,0,0
Similarly the direction cosines of y axis are 0,1, 0 and z axis are 0,0,1 respectively.

9. A line is uniquely determined if
1) It passes through a given point and has given direction
OR
2) It passes through two given points.

10. Two lines with direction ratios a_1, a_2, a_3 and b_1, b_2, b_3 respectively are perpendicular if:
$a_1b_1 + a_2b_2 + c_1c_2 = 0$

11. Two lines with direction ratios a_1, a_2, a_3 and b_1, b_2, b_3 respectively are parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

12. The lines which are neither intersecting nor parallel are called as skew lines. Skew lines are non coplanar i.e. they don't belong to the same 2D plane.

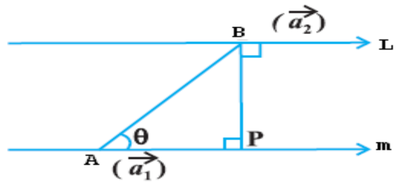


GE and DB are skew lines.

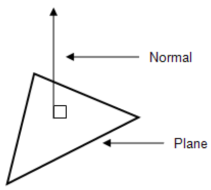
13. **Angle between skew lines** is the angle between two intersecting lines drawn from any point (preferably through the origin) parallel to each of the skew lines.

14. If two lines in space are intersecting then the shortest distance between them is zero.

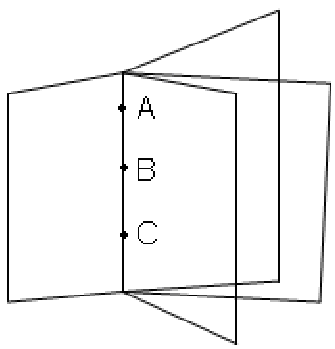
15. If two lines in space are parallel, then the shortest distance between them is the perpendicular distance.



16. The normal vector often simply called the "normal," to a surface; is a vector perpendicular to that surface.



17. If the three points are collinear, then the line containing those three points can be a part of many planes



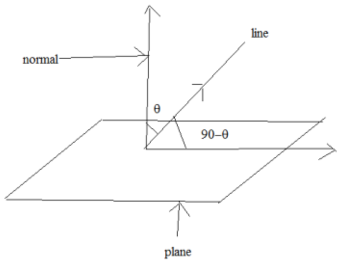
18. The angle between two planes is defined as the angle between their normals.

19. If the planes $A_1x+B_1y+C_1z+D_1=0$ and $A_2x+B_2y+C_2z+D_2=0$ are perpendicular to each other, then $A_1A_2+B_1B_2+C_1C_2=0$

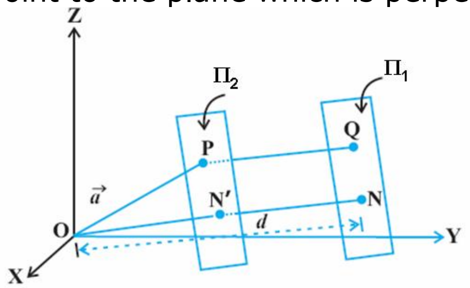
20. If the planes $A_1x+B_1y+C_1z+D_1=0$ and $A_2x+B_2y+C_2z+D_2=0$ are parallel, then

$$\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}$$

21. The angle between a line and a plane is the complement of the angle between the line and the normal to the plane.



22. Distance of a point from a plane is the length of the unique line from the point to the plane which is perpendicular to the plane.



Key Formulae

1. Direction cosines of the line L are connected by the relation
   $\ell^2+m^2+n^2=1$
2. If a, b, c are the direction ratios of a line and λ,m,n the direction cosines then,
   $$\ell=\pm\frac{a}{\sqrt{a^2+b^2+c^2}},\quad m=\pm\frac{b}{\sqrt{a^2+b^2+c^2}},\quad n=\pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

   The direction cosines of the line joining P(x_1,y_1,z_1) and Q(x_2,y_2,z_2) are
3. $$\frac{x_2-x_1}{PQ},\frac{y_2-y_1}{PQ},\frac{z_2-z_1}{PQ}$$
   where $PQ=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

   Vector equation of a line that passes through the given point whose position vector
4. is $\vec{a}$ and parallel to a given vector $\vec{b}$ is
   $\vec{r}=\vec{a}+\lambda\vec{b}$

   If coordinates of point A be (x_1,y_1,z_1) and Direction ratios of the line be a, b, c
5. Then, cartesian form of equation of line is :
   $$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

6.
If coordinates of point A be (x_1,y_1,z_1) and direction cosines of the line be ℓ,m,n
Then, cartesian equation of line is :

$$\frac{x-x_1}{\ell}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

7.
The vector equation of a line which passes throughtwo points whose position vectors are $\vec{a}$ and $\vec{b}$ is $\vec{r}=\vec{a}+\lambda(\vec{b}-\vec{a})$

8.
Cartesian equation of a line that passes through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

9. Angle θ between two lines L_1 and L_2 passing through origin and having direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 is

$$\cos\theta=\left|\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

Or $\sin\theta=\dfrac{\sqrt{(a_1b_2-a_2b_1)^2+(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2}}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$

10. Shortest distance between two skew lines L and m,
$\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ and $\vec{r}=\vec{a_2}+\mu\vec{b_2}$ is

$$d=\left|\frac{\vec{b_1}\times\vec{b_2}.(\vec{a_2}-\vec{a_1})}{|\vec{b_1}\times\vec{b_2}|}\right|$$

11. The shortest distance between the lines in Cartesian form

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1} \text{ and } \frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$$

is given by

$$d=\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}$$

12. Distance between parallel lines $\vec{r} = \vec{a_1} + \lambda\vec{b}$ and $\vec{r} = \vec{a_2} + \mu\vec{b}$ is

$$d = \left| \frac{\vec{b} \times (\vec{a_2} - \vec{a_1})}{|\vec{b}|} \right|$$

13. Equation of a plane which is at a distance d from the origin, and $\hat{n}$ is the unit vector normal to the plane through the origin in vector form is

$$\vec{r}.\hat{n} = d$$

14. Equation of a plane which is at a distance of d from the origin and the direction cosines of the normal to the plane as l, m, n is
$lx + my + nz = d$.

15. Equation of a plane perpendicular to a given vector $\vec{N}$ and passing through a given point $\vec{a}$ is $(\vec{r} - \vec{a}).\vec{N} = 0$

16. Equation of a plane perpendicular to a given line with direction ratios A, B, C and passing through a given point (x_1, y_1, z_1) is
$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$

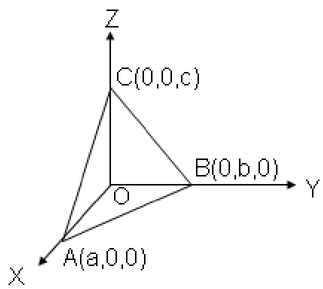
17. Equation of a plane passing through three non-collinear points in vector form is given as
$(\vec{r} - \vec{a}).[(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$

18. Equation of a plane passing through three non collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) in Cartesian form is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

19. Intercept form of equation of a plane
$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ where a, b and c are the intercepts on x, y and z-axes

respectively.



20. Any plane passing thru the intersection of two planes $\vec{r}\,.\,\overrightarrow{n_1} = d_1$ and $\vec{r}\,.\,\overrightarrow{n_2} = d_2$ is given by,

$$\vec{r}.\left(\overrightarrow{n_1} + \lambda\overrightarrow{n_2}\right) = d_1 + \lambda d_2$$

21. Cartesian Equation of plane passing through intersection of two planes

$$(A_1x + B_1y + C_1z - d_1 + \lambda(A_2x + B_2y + C_2z - d_2) = 0$$

22. The given lines $\vec{r} = \overrightarrow{a_1} + \lambda\overrightarrow{b_1}$ and $\vec{r} = \overrightarrow{a_2} + \mu\overrightarrow{b_2}$ are coplanar if and only

$$\left(\overrightarrow{a_2} - \overrightarrow{a_1}\right).\left(\overrightarrow{b_1} \times \overrightarrow{b_2}\right) = 0$$

23. Let (x_1,y_1,z_1) and (x_2,y_2,z_2) be the coordinates of the points M and N respectively.
Let a_1, b_1, c_1 and a_2, b_2, c_2 be the direction ratios of $\overrightarrow{b_1}$ and $\overrightarrow{b_2}$ respectively.
The given lines are coplanar if and only if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

24. If $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ are normals to the planes
$\vec{r}.\overrightarrow{n_1} = d_1$ and $\vec{r}.\overrightarrow{n_2} = d_2$ and θ is the angle between the normals drawn from some common point then

$$\cos\theta = \left|\frac{\overrightarrow{n_1}.\overrightarrow{n_2}}{\left|\overrightarrow{n_1}\right|\left|\overrightarrow{n_2}\right|}\right|$$

25. Let θ is the angle between two planes $A_1x+B_1y+C_1z+D_1=0$, $A_2x+B_2y+C_2z+D_2=0$
The direction ratios of the normal to the planes are A_1, B_1, C_1 and A_2, B_2,, C_2.

$$\cos\theta = \left|\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}}\right|$$

26. The angle θ between the line and the normal to the plane is given by

$$\cos\theta=\left|\frac{|\vec{b}.\vec{n}|}{|\vec{b}||\vec{n}|}\right|$$

or $\sin\ \phi=\left|\frac{|\vec{b}.\vec{n}|}{|\vec{b}||\vec{n}|}\right|$ where $\phi = 90^{o} - \phi$

27. Distance of point P with position vector $\vec{a}$ from a plane $\vec{r}.\vec{N} = d$ is $\frac{|\vec{a}.\vec{N}-d|}{|\vec{N}|}$

where $\vec{N}$ is the normal to the plane

28. The length of perpendicular from origin O to the plane $\vec{r}.\vec{N} = d$ is $\frac{|d|}{|\vec{N}|}$

where $\vec{N}$ is the normal to the plane.