Class XII: Mathematics

Chapter 4: Determinants

Chapter Notes

Top Definitions

1. To every square matrix A $=[a_{ij}]$ a unique number (real or complex) called determinant of the square matrix A can be associated. Determinant of matrix A is denoted by det(A) or |A| or Δ.
2. A determinant can be thought of as a function which associates each square matrix to a unique number (real or complex).

   $f: M \to K$ is defined by $f(A) = k$ where $A \in M$ the set of square matrices and $k \in K$ set of numbers(real or complex)
3. Let A = [a] be the matrix of order 1, then determinant of A is defined to be equal to a.
4. Determinant of order 2

   $$\text{If } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{then, } |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$
5. Determinant of order 3

   $$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ then}$$

   $$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$
6. Minor of an element a_{ij} of a determinant is the determinant obtained by deleting its ith row and jth column in which element a_{ij} lies. Minor of an element a_{ij} is denoted by M_{ij}.

7. Cofactor of an element $a_{ij,}$ denoted by A_{ij} is defined by
   $A_{ij} = (-1)^{i+j.} M_{ij}$ where M_{ij} is the minor of a_{ij}.

8. The adjoint of a square matrix $A=[a_{ij}]$ is the transpose of the cofactor matrix $[Aij]_{n\times n}$.
9. A square matrix A is said to be singular if $|A| = 0$

10. A square matrix A is said to be non – singular if $|A| \neq 0$

11. If A and B are nonsingular matrices of the same order, then AB and BA are also nonsingular matrices of the same order.

12. The determinant of the product of matrices is equal t product of the respective determinants, that is, $|AB| = |A|\ |B|$, where A and B are square matrices of the same order.
13. A square matrix A is invertible i.e its inverse exists if and only if A is nonsingular matrix. Inverse of matrix A if exists is given by

$$A^{-1} = \frac{1}{|A|}(\text{adj}\,A)$$

14. A system of equations is said to be consistent if its solution (one or more) exists.
15. A system of equations is said to be inconsistent if its solution does not exist.

Top Concepts

1. A determinant can be expanded along any of its row (or column). For easier calculations it must be expanded along the row (or column) containing maximum zeros.

2. If A=kB where A and B are square matrices of order n, then $|A|=k^n |B|$ where n =1,2,3.

   Properties of Determinants

3. **Property 1** Value of the determinant remains unchanged if its row and columns are interchanged. If A is a square matrix, the det (A) = det (A′), where A′ = transpose of A.
4. **Property 2** If two rows or columns of a determinant are interchanged, then the sign of the determinant is changed. Interchange of rows and columns is written as $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$
5. **Property 3:** If any two rows (or columns) of a determinant are identical, then value of determinant is zero.
6. **Property 4**: If each element of a row (or a column) of a determinant is multiplied by a constant k, then its value get multiplied by k. If Δ_1 is the determinant obtained by applying $R_i \rightarrow kR_i$ or $C_i \rightarrow kC_i$ to the determinant Δ, then $\Delta_1 = k\Delta$. So .if A is a square matrix of order n and k is a scalar, then $|kA|=k^n|A|$. This property enables taking out of common factors from a given row or column.
7. **Property 5:** If in a determinant, the elements in two rows or columns are proportional, then the value of the determinant is zero. For example.

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ ka_1 & ka_2 & ka_3 \end{vmatrix} = 0 \text{ (rows } R_1 \text{ and } R_3 \text{ are proportional)}$$

8. **Property 6:** If the elements of a row (or column) of a determinant are expressed as sum of two terms, then the determinant can be expressed as sum of two determinants.

9. **Property 7:** If to any row or column of a determinant, a multiple of another row or column is added, the value of the determinant remains the same i.e the value of the determinant remains same on applying the operation $R_i \to R_i + kR_j$ or $C_i \to C_i + k C_j$

10. If more than one operation like $R_i \to R_i + kR_j$ is done in one step, care should be taken to see that a row that is affected in one operation should not be used in another operation. A similar remark applies to column operations.

11. Since area is a positive quantity, so the absolute value of the determinant is taken in case of finding the area of the triangle.

12. If area is given, then both positive and negative values of the determinant are used for calculation.

13. The area of the triangle formed by three collinear points is zero.

14. Minor of an element of a determinant of order $n(n \geq 2)$ is a determinant of order $n - 1$.

15. Value of determinant of a matrix A is obtained by sum of product of elements of a row (or a column) with corresponding cofactors. For example $|A| = a_{11}A_{11}+a_{12}A_{12}+a_{13}A_{13}$

16. If elements of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero. For example $a_{11}A_{21}+a_{12}A_{22}+a_{13}A_{23} = 0$

17. If A is a nonsingular matrix of order n then $|adj.A|=|A|^{n-1}$

18. Determinants can be used to find the area of triangles if its vertices are given

19. Determinants and matrices can also be used to solve the system of linear equations in two or three variables.

20. System of equations
$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

can be written as A X = B, where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{and} B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Then matrix $X = A^{-1}B$ gives the unique solution of the system of equations if |A| is non zero and A^{-1} exists.

Top Formulae

1. Area of a Triangle with vertices (x_1, y_1), (x_2, y_2) & (x_3, y_3) is

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

2. Determinant of a matrix $A = [a_{ij}]_{1 \times 1}$ is given by $|a_{11}| = a_{11}$

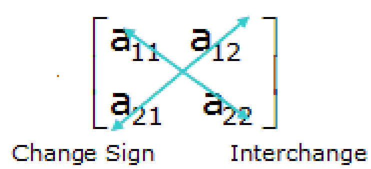
3. $$\text{If } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{then, } |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ then}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

4. Cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} \cdot M_i$

5. $\text{If } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then adj.A =



$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{then adj} A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

where, A_{ij} are cofactors of a_{ij}

6. $|AB| = |A|\ |B|$,

7. $A^{-1} = \frac{1}{|A|}(\text{adj} A)$ where $|A| \neq 0$.

8. $|A^{-1}| = \frac{1}{|A|}$ and $(A^{-1})^{-1} = A$

9. Unique solution of equation AX = B is given by $X = A^{-1}B$, where $|A| \neq 0$.

10. For a square matrix A in matrix equation AX = B,

i. $|A| \neq 0$, there exists unique solution.

ii. $|A| = 0$ and (adj A) B $\neq 0$, then there exists no solution.

iii. $|A| = 0$, and (adj A) B = 0, then system may or may not be consistent.